

Monetary Theory

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1 Dynamic Programming

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1.1 Poisson processes

Definition. A stochastic process $X = \{X(t), t \in \mathcal{T}\}$ is a collection of random variables indexed by a set.

1. We say that $X(t)$ has *independent increments* when for every $t_0 < \dots < t_n$, $X(t_1) - X(t_0)$ and $X(t_3) - X(t_2)$ are independent.
2. We say that $X(t)$ has *stationary increments* when, for every s , $X(t + s) - X(t)$ has the same distribution.

Definition. A stochastic process $N(t)$ is a *counting process* when

1. $N(t) \in \mathbb{N}$ wp1
2. For all $s < t$, $N(s) \leq N(t)$ wp1
3. For all $s < t$, $N(t) - N(s)$ is the number of events in $(s, t]$.

Definition. A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is $o(\Delta)$ if $\lim_{\Delta \rightarrow 0} \frac{f(\Delta)}{\Delta} = 0$.

Definition. A counting process $\{N(t) : t \geq 0\}$ is said to be a *Poisson process* with rate $\alpha > 0$ if

1. $N(0) = 0$.
2. The process has independent and stationary increments
3. As $\Delta \rightarrow 0$,
 - a. $\mathbb{P}(N(\Delta) = 1) = \alpha\Delta + o(\Delta)$

$$b. \mathbb{P}(N(\Delta) = 0) = (1 - \alpha)\Delta + o(\Delta)$$

$$c. \mathbb{P}(N(\Delta) \geq 2) = o(\Delta)$$

Proposition. A counting process N is a Poisson process if and only if

1. $N(0) = 0$
2. N has independent increments
3. The number of arrivals in any interval of length Δ is Poisson with mean $\alpha\Delta$ (notice that this implies stationary increments), that is, $\mathbb{P}(N(t + \Delta) - N(t) = n) = \frac{(\alpha\Delta)^n e^{-\alpha\Delta}}{n!} = p_n(\Delta)$.

Equivalence proof omitted

1.2 Interarrival times

For each n , let X_n be the time elapsed between the $(n - 1)$ th and the n th arrival.

Example. Assume meetings follow a Poisson process, the distribution of X_1 is

$$\mathbb{P}(X_1 \leq t) = 1 - \mathbb{P}(X_1 > t) = 1 - p_0(t) = 1 - e^{-\alpha t}$$

that is, X_1 has an exponential distribution with parameter α . Also,

$$\begin{aligned} \mathbb{P}(X_2 \leq t + s | X_1 = s) &= 1 - \mathbb{P}(X_2 > t + s | X_1 = s) = 1 - \mathbb{P}(N(t + s) - N(t) = 0 | X_1 = s) \\ &= 1 - \mathbb{P}(N(t + s) - N(t) = 0) = 1 - p_0(s) = 1 - e^{-\alpha s} \end{aligned}$$

Proposition. The interarrival times $\{X_m\}_m$ of a Poisson process with rate α are identically distributed exponential random variables with mean $1/\alpha$.

1.3 Search in continuous time with Poisson processes

Two presentations of a continuous time McCall model, in decreasing order of formality

1. We pretend time is discrete: $t = 0, \Delta, 2\Delta, \dots$. We also apply a discount factor $\beta(\Delta)$ for a period of length Δ . The worker is assumed to start unemployed. Let n be the (random) number of job offers received in a period of length Δ , and $p_n(\Delta)$ be its distribution. A job offer is an *iid* draw from an exogenous distribution F , with support $[0, \bar{w}]$, of wages. Let $\omega_n = \max\{\omega_1, \dots, \omega_n\}$ be the best offer received, conditional on having received n offers in the current period, and $G(x, n) = \mathbb{P}(\omega_n \leq x)$.

If employed, the worker receives $\omega\Delta$ each period, while he receives $b\Delta$ when unemployed. The value of an employed worker at wage w is

$$v(w) = w\Delta + \beta(\Delta)v(w)$$

The value of an unemployed worker is

$$U = b\Delta + \beta(\Delta) \left[p_0(\Delta)U + \sum_{n=1}^{\infty} p_n(\Delta) \int \max\{v(x), U\} dG(x, n) \right]$$

Now, assume offers follow a Poisson process and that $\beta(\Delta) = e^{-r\Delta}$. Now,

$$\lim_{\Delta \rightarrow 0} \frac{1}{\Delta} (1 - e^{-r\Delta})v(w) = w \implies rv(w) = w$$

and

$$\begin{aligned} U &= b\Delta + e^{-r\Delta} \left[\underbrace{(1 - \alpha\Delta + o(\Delta))}_{p_0(\Delta)} U + \underbrace{(\alpha\Delta + o(\Delta))}_{p_1(\Delta)} \int \max(v(w), U) dF(w) + o(\Delta) \right] \\ \implies rU &= b + \alpha \underbrace{\int \max\{v(w) - U, 0\} dF(w)}_J \end{aligned}$$

2. Look at the continuous time problem directly, and let τ be the interarrival time. Then,

$$\begin{aligned} U &= \mathbb{E}_{\tau} \left[\int_t^{t+\tau} e^{-r(s-t)} b ds + e^{-r\tau} (J + U) \right] = \int_0^{\infty} \left[\int_t^{t+\tau} e^{-r(s-t)} b ds + e^{-r\tau} (J + U) \right] \alpha e^{-\alpha\tau} d\tau \\ &= \frac{b + \alpha(J + U)}{r + \alpha} \implies rU = b + \alpha J \end{aligned}$$

as before.

2 Diamond '84

September 9, 2014

Coordination problems can reintroduce inefficient unemployment (and deficient demand) even with rational expectations and full price flexibility, when the Walrasian auctioneer is replaced by bilateral trades.

2.1 Environment

- Continuous time with infinite horizon.
- Population: $i \in [0, 1]$.
- Commodities: 1 indivisible good.
- Production technology: a Poisson process with rate $a > 0$. Cost $c \sim G(\cdot)$ of production, with support $[\underline{c}, \bar{c}]$.
- Simplifications:

- Inventory capacity constraint: Agents can only hold one unit of the good at any given time.
- Agents get no utility from consuming own output.
- “No credit”
- Mechanism of exchange: an agent that wants to exchange finds a trading partner according to a Poisson process of parameter $b > 0$.
- Preferences: consuming one unit gives instantaneous utility $u > \underline{c}$. An agent's utility function looks like

$$\mathbb{E}_0 \left[\sum_{i=1}^{\infty} e^{-rt_i} \mathbb{1}(t_i \in \Pi_u) u - \mathbb{1}(t_i \in \Pi_c) c_{t_i} \right]$$

where Π_u and Π_c are the set of Poisson times at which the agent consumes and produces.

2.2 Exchange

Let e be the measure of agents in the exchange state (it is also the total supply of coconuts). Also, because of the inventory capacity constraint and homogeneity of the good, the only exchange that makes sense in equilibrium is a coconut-for-coconut swap.

The arrival rate in the exchange island is $b : [0, 1] \rightarrow \mathbb{R}_+$, with $b'(e) > 0$ and $b(0) = 0$: it is easier to find partners if there are more people searching (but the model is competitive, no one internalizes their effect on the b function).

3 Some bargaining

September 30, 2014

3.1 Kalai '77 (Proportional Bargaining)

Punchline here is that each agent gets a constant proportion of what is being bargained over.

Definition. Given a set of players $\mathcal{I} = \{1, \dots, N\}$, a 0-normalized *bargaining game* is a set S such that

- a. $S \subseteq \mathbb{R}^{\mathcal{I}}$, S is convex and compact
- b. S is comprehensive: for every $x \in S$ and $y \in \mathbb{R}_+^{\mathcal{I}}$ such that $y \leq x$, then $y \in S$.
- c. $\exists s \in S$ such that $s_i > 0$ for each $i \in \mathcal{I}$.

Notation. Let $\mathcal{B}$ be the set of all such bargaining games.

Definition. A *bargaining solution* is a function $f : \mathcal{B} \rightarrow \mathbb{R}_+^{\mathcal{I}}$

Axioms

(PAR) Pareto: There is no $y \in S$ such that $y_i > f_i(S)$ for each i .

(SIR) Strong individual rationality: $f_i(S) > 0$ for all i .

(HOM) Homogeneity: for any $c \in \mathbb{R}_{++}$, $f(cS) = cf(S)$ (notice that we only require invariance to scalar transformations that have the same scalar for each player).

and

(MON) Monotonicity: a solution f is monotonic if for any $S, T \in \mathcal{B}$, $S \subseteq T$, we have that $f_i(S) \leq f_i(T)$.

(SBS) Step-by-step negotiations: whenever $U, S \in \mathcal{B}$, $U \subseteq S$, $(S - f(U)) \cap \mathbb{R}_+^{\mathcal{J}} \in \mathcal{B}$, we have that $f(S) = f(U) + f((S - f(U)) \cap \mathbb{R}_+^{\mathcal{J}})$.

(IIA) Independence of irrelevant alternatives.

Definition. A bargaining solution is said to be *proportional* if there exists a vector $p \in \mathbb{R}_{++}^{\mathcal{J}}$ such that for every $S \in \mathcal{B}$, $f(S) = pV(S)$, where $V(S) = \max\{v \in \mathbb{R} : pv \in S\}$.

Remark Given p , $\theta = \frac{p}{\sum_i p_i}$ is the vector of bargaining powers. Also, $f_i(S) = \theta_i V(S)$ and, hence, for every S , we have that

$$\frac{f_i(S)}{f_j(S)} = \frac{\theta_i}{\theta_j}$$

Theorem. (Kalai)

1. A bargaining solution $f : \mathcal{B} \rightarrow \mathbb{R}_+^{\mathcal{J}}$ satisfies (PAR), (SIR), (HOM), and (MON) if and only if it is proportional.
2. A bargaining solution $f : \mathcal{B} \rightarrow \mathbb{R}_+^{\mathcal{J}}$ satisfies (PAR), (SIR), (HOM), and (SBS) if and only if it is proportional.

Example 3.1. Splitting a cake of size 1 between two players

$$\begin{aligned} \theta [\mu_2(q) - d_2] &= (1 - \theta) [\mu_1(q) - d_1] && \text{under Kalai} \\ \theta(q) [\mu_2(q) - d_2] &= (1 - \theta(q)) [\mu_1(q) - d_1] && \text{under Nash} \end{aligned}$$

where

$$\theta(q) = \frac{\theta \mu'_1(q)}{\theta \mu'_1(q) - (1 - \theta) \mu'_2(q)}$$

is the distorted power.

Notice that Nash and Kalai coincide when $\mu_1(q) = q$ and $\mu_2(q) = 1 - q$.

Example 3.2. *General good and special good*

$$\text{Nash: } (q^N, p^N) = \arg \max_{q,p} (\mu(q) - p - d_1)^\theta (-c(q) + p - d_2)^{1-\theta}$$

$$\text{subject to } \begin{cases} q \geq 0 \\ -z \leq p \leq z \\ \mu(q) - p - d_1 \geq 0 \\ -c(q) + p - d_2 \geq 0 \end{cases}$$

$$\text{Kalai: } (q^K, p^K) = \arg \max_{q,p} (\mu(q) - c(q))$$

$$\text{subject to } \begin{cases} (1-\theta)(\mu(q) - p - d_1) = \theta(-c(q) + p - d_2) \\ q \geq 0 \\ -z \leq p \leq z \\ \mu(q) - p - d_1 \geq 0 \\ -c(q) + p - d_2 \geq 0 \end{cases}$$

We can ignore the nonnegativity on q and the lower bound on p because of reasons, and guess that the other nonnegativities won't bind either. Then,

$$\mathcal{L} = \mu(q) - c(q) - \mu \{ \theta(c(q) + d_2) + (1-\theta)(\mu(q) - d_1) - z \}$$

If $\mu = 0$, then $q = q^*$. On the other hand, when $\mu > 0$, we have $z = \theta(c(q) + d_2) + (1-\theta)(\mu(q) - d_1)$, the $g(q)$ from the Nash solution, only with the original weights, not the distorted ones. Hence,

$$q^K = \begin{cases} q^* & \text{if } z \geq \theta(c(q) + d_2) + (1-\theta)(\mu(q) - d_1) \\ \theta(c(q) + d_2) + (1-\theta)(\mu(q) - d_1) & \text{otherwise} \end{cases}$$

$$p^K = \begin{cases} \theta(c(q) + d_2) + (1-\theta)(\mu(q) - d_1) & \text{if } z \geq \theta(c(q) + d_2) + (1-\theta)(\mu(q) - d_1) \\ z & \text{otherwise} \end{cases}$$

Nash distorts the weights with marginal utilities. Kalai keeps them fixed, so q falls short of q^* only when the liquidity constraint binds.

Obs *on the (HOM) axiom*

We want to split y between two players, let x_i be the share to i , d_i a disagreement point, and p_i the Kalai weights. Let $u_i(x_i) = A_i x_i$. The Kalai solution is characterized by $p_2(u_1(x_1) - d_1) = p_1(u_2(x_2) - d_2)$ and $x_1 + x_2 = y$. Calculations yield

$$x_1 = \frac{p_1 A_2}{p_1 A_2 + p_2 A_1} y; \quad x_2 = \frac{p_2 A_1}{p_1 A_2 + p_2 A_1} y$$

4 Second Generation Models of Money

Trejos and Wright (1995), Shi (1995)

Goal is to introduce divisibility in goods so that we can have prices.

Environment is the same as the simplest Kiyotaki-Wright: nonstorable goods and storable coins. The new elements are

- Agents have a utility and a cost function:
 - $u(0) = c(0) = 0$
 - $u' > 0, c' > 0$
 - $u'' \leq 0, c'' \geq 0$, at least one of those strict
 - $\exists q : u(q) = c(q)$ and $\exists q^* : u'(q^*) = c'(q^*)$.
- All meetings are single-coincidence meetings
- Gains from trade: q is determined by Kalai bargaining (not necessary to say how many coins because it's always for one coin) with θ being the weight of a buyer.

4.1 Bellman Equations

Let $V_k(t)$ be the value of holding k coins ($k \in \{0, 1\}$).

$$rV_0(t) = \beta\sigma M\Pi_1(t)\pi_0(t)(-c(q(t)) + \underbrace{V_1(t) - V_0(t)}_{=\Gamma(t)}) + \dot{V}_0(t)$$

$$rV_1(t) = \beta\sigma(1-M)\Pi_0(t)\pi_1(t)(u(q(t)) + V_0(t) - V_1(t)) + \dot{V}_1(t)$$

Because of Kalai bargaining, $q(t)$ satisfies $(1 - \theta)[u(q(t)) + V_0(t) - V_1(t)] = \theta[-c(q(t)) + V_1(t) - V_0(t)]$. Also,

$$\pi_1(t) \begin{cases} = 1 & \text{if } \Gamma(t) > u(q(t)) \\ \in [0, 1] & \text{if } \Gamma(t) = u(q(t)) \\ = 0 & \text{if } \Gamma(t) < u(q(t)) \end{cases}$$

Notice that $\Gamma(t) = g(q) = \theta c(q(t)) + (1 - \theta)u(q(t))$, so that $\Gamma(t)$, the value of the coin, is the z from Example 2. Also, the gains from trade are distributed as $u(q(t)) - \Gamma(t) = \theta(u(q(t)) - c(q(t)))$ and $-c(q(t)) + \Gamma(t) = (1 - \theta)(u(q(t)) - c(q(t)))$. Hence, we can write

$$\pi_1(t) \begin{cases} = 1 & \text{if } \theta(u(q(t)) - c(q(t))) > 0 \\ \in [0, 1] & \text{if } \theta(u(q(t)) - c(q(t))) = 0 \\ = 0 & \text{if } \theta(u(q(t)) - c(q(t))) < 0 \end{cases}$$

We can also replace the $\Gamma(t)$ terms in the Bellman equations to get

$$\begin{aligned} rV_0(t) &= \beta\sigma M\Pi_1(t)\pi_0(t)(1-\theta)\{u(q(t)) - c(q(t))\} + \dot{V}_0(t) \\ rV_1(t) &= \beta\sigma(1-M)\Pi_0(t)\pi_1(t)(1-\theta)\{u(q(t)) - c(q(t))\} + \dot{V}_1(t) \end{aligned}$$

Now, subtracting,

$$\begin{aligned} r\Gamma(t) &= \beta\sigma\{(1-M)\Pi_0(t)\pi_1(t)\theta(u(q(t)) - c(q(t))) - M\Pi_1(t)\pi_0(t)(1-\theta)(u(q(t)) - c(q(t)))\} + \\ &\quad + \dot{\Gamma}(t) \end{aligned} \tag{A1}$$

or

$$\begin{aligned} rg(q(t)) &= \beta\sigma\{(1-M)\Pi_0(t)\pi_1(t)\theta(u(q(t)) - c(q(t))) - M\Pi_1(t)\pi_0(t)(1-\theta)(u(q(t)) - c(q(t)))\} + \\ &\quad + g'(q(t))\dot{q}(t) \end{aligned}$$

4.2 Equilibrium

Definition. A symmetric equilibrium is a path $\left\{\{\pi_k(t), \Pi_k(t), V_k(t)\}_{k=0,1}, q(t)\right\}_{t \geq 0}$ such that everything in the previous section is satisfied and $\pi_k(t) = \Pi_k(t)$.

Analysis Because of all the substitutions, if we were to guess a pure strategy monetary equilibrium, it all boils down to finding a path for q that satisfies the differential equation (A1) and the conditions for π_k and such that $0 \leq q(t) \leq \bar{q}$.

Now, rewrite (A1) as $g'(q)\dot{q} = T(q)$. A pure strategy steady state monetary equilibrium is just a $q \in [0, \bar{q}]$ such that $\dot{q} = 0$ and hence

$$0 = T(q) = ru(q) - (r\theta + \beta\sigma(\theta - M))(u(q) - c(q))$$

Also notice that $T(0) = 0$ and that $T(\bar{q}) > 0$.

Is $q = q^*$? Generically no.

Note Is there a $\theta \in [0, 1]$ that induces $q = q^*$ in the steady-state monetary equilibrium? Just plug q^* in $T(q) = 0$ and solve for θ to get

$$\theta_* = \frac{ru(q^*) + \beta\sigma M(u(q^*) - c(q^*))}{(r + \beta\sigma)(u(q^*) - c(q^*))}$$

Exercise. Find $(q^0, M^0) = \arg \max_{q, M} \{MV_1 + (1-M)V_0\}$.

Reading Trejos and Wright (2013)

5 Third Generation Models of Money

October 7, 2014

5.1 Environment

- Time is discrete, with infinite horizon
- Each period has two subperiods
- Population is constant: agents $i \in [0, 1]$ ex-ante identical
- There are 2 nonstorable commodities at each date
 - General good: everyone likes them and can produce them equally well
 - Special good: subject to a double coincidence friction
- People have a labor endowment that they can use to produce those two goods. Also some asset endowments
- Preferences given by

$$\mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u(q_t, h_t, Q_t, H_t) \right]$$

where q is quantities and h is effort, capitals are for the second subperiods (general good), lowercase for the first.

- $\beta \in (0, 1)$
- $u(q, h, Q, H) = u(q) - c(h) + U(Q) - C(H)$. Further, either $U(Q) = Q$ or $C(H) = H$ (what's important is quasi-linearity). We'll make it quasi-linear in labor here.
- $u' > 0$, $u'' < 0$ and $c' > 0$, $c'' > 0$, $U' > 0$, $U'' \leq 0$.
- $c(0) = u(0) = 0$, there exists $\bar{q}$ such that $u(\bar{q}) = c(\bar{q})$, and there exists q^* such that $u'(q^*) = c'(q^*)$, the quantity that maximizes static gains from trade in each meeting. There also exists an x^* such that $U'(x^*) = 1$, so that each agent has an optimal quantity for the general good as well.
- Market structure
 - In the first subperiod, there is random search and bilateral trade just like in Trejos-Wright-Shi, except there is no bound on money holdings.
 - * Conditional on meeting somebody, there is a σ probability of a single coincidence where the agent buys, σ of single coincidence where the agent sells, δ probability of double coincidence, and $1 - 2\sigma - \delta$ probability of no coincidence. α is the meeting rate.
 - In the second subperiod, agents trade against their budget sets.

- Asset structure
 - Fiat money: perfectly divisible, storable, no limit on holdings $m \in \mathbb{R}_+$
 - Let M_t be the stock of money (in a while, we'll have $M_{t+1} = M_t + T_t$ for some transfers/taxes T_t)
- Information structure: Anonymity (no enforcement or commitment, no record-keeping), so all trades are *quid-pro-quo*.
- Equilibrium concept

5.2 Bellman Equations

An agent has value $V_t(m)$ at the beginning of period t (before the search subperiod) and $W_t(m)$ before the competitive subperiod of period t . Then, if ϕ_t is general goods per dollar (reciprocal of the price) and Y is “effort” (or that the upper bound on labor is not binding in equilibrium)

$$W_t(m) = \max_{X, Y, m'} \{U(X) - Y + \beta V_{t+1}(m')\}$$

$$\text{subject to } \begin{cases} X + \phi_t m' = Y + \phi_t m + \phi_t T_t \\ X \geq 0, Y \geq 0 \\ m' \geq 0 \end{cases}$$

The constraints can be rewritten as $Y = X + \phi_t(m' - m) - \phi_t T_t \geq 0$. Hence,

$$W_t(m) = \max_{X, m'} \{U(X) - X + \phi_t m + \phi_t T_t - \phi_t m' + \beta V_{t+1}(m')\}$$

$$\text{subject to } X \geq 0, X + \phi_t(m' - m) - \phi_t T_t \geq 0, m' \geq 0$$

Because of separability and quasi-linearity, we can separate the decision of m' from that of X to get

$$W_t(m) = \phi_t m + \phi_t T_t + U(x^*) - x^* + \max_{m'} \{-\phi_t m' + \beta V_{t+1}(m')\} \quad (*)$$

subject to the same constraints as before. Notice that $W_t(m) = \phi_t m + W_t(0)$ so W_t is affine in money holdings.

Every agent chooses the same m' so there is no need to keep track of distributions.

In the search market, suppose that a buyer with m dollars meets a seller with $\tilde{m}$ dollars at t and let the terms of trade be $(q_t(m, \tilde{m}), d_t(m, \tilde{m}))$: amount of goods for amount of dollars. Let

$F_t(m)$ be the *cdf* of money holdings and $B_t(m, \tilde{m})$ be the result of barter in a double coincidence meeting

$$\begin{aligned} V_t(m) = & \alpha\sigma \int u[q_t(m, \tilde{m})] + W_t[m - d_t(m, \tilde{m})] dF_t(\tilde{m}) \\ & + \alpha\sigma \int -c[q_t(\tilde{m}, m)] + W_t[m + d_t(\tilde{m}, m)] dF_t(\tilde{m}) \\ & + \alpha\delta \int B_t(m, \tilde{m}) dF_t(\tilde{m}) \\ & + (1 - 2\alpha\sigma - \alpha\delta)W_t(m) \end{aligned}$$

5.3 Bargaining

Two situations: double or single coincidence meetings. In δ -meetings, we assume Nash bargaining with equal weights. In σ -meetings, assume Nash bargaining with the buyer having bargaining power θ .

5.3.1 δ -meetings

$$\max_{q_1, q_2, d} \left[\underbrace{u(q_1) - c(q_2) + W_t(m + d)}_{\text{gains from trade}} - \underbrace{W_t(m)}_{\text{threat point}} \right] [u(q_2) - c(q_1) + W_t(\tilde{m} - d) - W_t(\tilde{m})]$$

subject to $-m \leq d \leq \tilde{m}$ and both brackets being positive. Now, notice that $W_t(m + d) - W_t(m) = \phi_t d$. Hence, the Nash bargaining problem is just

$$\max_{q_1, q_2, d} [u(q_1) - c(q_2) + \phi_t d] [u(q_2) - c(q_1) - \phi_t d]$$

Lemma. $B_t(m, \tilde{m}) = u(q^*) - c(q^*) + W_t(m)$: in double-coincidence meetings, agents produce the static optimum and don't exchange money holdings.

5.3.2 σ -meetings

The problem is

$$\max_{q, d} \underbrace{[u(q) + W_t(m - d) - W_t(m)]}_{\text{Gains to the buyer}}^\theta [-c(q) + W_t(\tilde{m} + d) - W_t(m)]^{1-\theta}$$

subject to nonnegativity and $-\tilde{m} \leq d \leq m$. As before, we can use quasi-linearity to get

$$\max_{q, d} [u(q) - \phi_t d] [-c(q) + \phi_t d]$$

subject to nonnegativity and $d \leq m$.

Lemma. Let $g(q) = \frac{\theta u'(q)c(q) + (1-\theta)c'(q)u(q)}{\theta u'(q) + (1-\theta)c'(q)}$. Then, Nash bargaining gives

$$q_t(m) = \begin{cases} q^* & \text{if } g(q^*) \leq \phi_t m \\ \hat{q}_t(m) & \text{otherwise} \end{cases}; \quad d_t(m) = \begin{cases} \frac{g(q^*)}{\phi_t} & \text{if } g(q^*) \leq \phi_t m \\ m & \text{otherwise} \end{cases}$$

where $g(\hat{q}_t(m)) = \phi_t m$.

Also,

Obs $\phi_t d_t = g(q_t(m))$

5.4 Euler Equation for money holdings

Now, we can plug back all of this into the value function:

$$\begin{aligned} V_t(m) &= \alpha \sigma(u(q_t(m)) + W_t(m - d_t(m))) \\ &\quad + \alpha \sigma \int -c(q_t(\tilde{m})) + W_t(m + d_t(\tilde{m})) dF_t(\tilde{m}) \\ &\quad + \alpha \delta(u(q^*) - c(q^*) + W_t(m)) \\ &\quad + (1 - 2\alpha\sigma - \alpha\delta)W_t(m) \end{aligned}$$

Or, subtracting $W_t(m)$'s

$$\begin{aligned} V_t(m) &= \alpha \sigma(u(q_t(m)) - g(q_t(m))) \\ &\quad + \alpha \sigma \int -c(q_t(\tilde{m})) + g(q_t(\tilde{m})) dF_t(\tilde{m}) \\ &\quad + \alpha \delta(u(q^*) - c(q^*)) + W_t(m) \end{aligned}$$

Rewrite as

$$V_t(m) = \alpha \sigma \Gamma_t^b(m) + \phi_t m + k_t$$

where $k_t = \alpha \sigma \Gamma_t^s + \alpha \delta(u(q^*) - c(q^*)) + W_t(0) = V_t(0)$

We want to get an Euler equation from (*). The part that depends on m' is

$$\max_{m'} \left\{ -\phi_t m' + \beta \left[\alpha \sigma \Gamma_{t+1}^b(m') + \phi_{t+1} m' \right] \right\}$$

or

$$\max_{m'} \left\{ \beta \alpha \sigma \Gamma_{t+1}^b(m') + (\beta \phi_{t+1} - \phi_t) m' \right\}$$

Obs In equilibrium, $\beta\phi_{t+1} - \phi_t \leq 0$. In particular, $\beta \leq \frac{\phi_t}{\phi_{t+1}} = 1 + \pi_t$. Many ways to see this

1. For $m' \geq m_t^* (= \frac{g(q^*)}{\phi_t})$, Γ_{t+1}^b is flat, so that there would be no solution if the condition is false.
2. Deflation would violate a transversality condition.

Lemma. 1. If $\theta \leq 1$, then $m_{t+1} < m_{t+1}^*$, even if $\beta\phi_{t+1} = \phi_t$.

2. If $\theta = 1$ and $\beta\phi_{t+1} - \phi_t < 0$, then $m_{t+1} < m_{t+1}^*$. Further, $m_{t+1} \rightarrow m_{t+1}^*$ if $\beta\phi_{t+1} - \phi_t \rightarrow 0$.

The question is what the slope in the gains from trade is (from bringing an extra dollar). In particular, whether the maximum is attained before or after Γ^b starts being flat. Notice that

$$\begin{aligned}\Gamma_t^b(m) &= u(q_t(m)) - g(q_t(m)) \\ \Rightarrow \frac{\partial \Gamma_t^b(m)}{\partial m} &= (u'(q_t(m)) - g'(q_t(m))) \frac{\partial q_t(m)}{\partial m} \\ &= (u'(q_t(m)) - g'(q_t(m))) \frac{\phi_t}{g'(q_t(m))} \\ &= \left(\frac{u'(q_t(m))}{g'(q_t(m))} - 1 \right) \phi_t\end{aligned}$$

Now,

$$g'(q) = \frac{u'(q)c'(q)(\theta u'(q) + (1-\theta)c'(q)) + \theta(1-\theta)(u(q) - c(q))(u'(q)c''(q) - c'(q)u''(q))}{(\theta u'(q) + (1-\theta)c'(q))^2} > 0$$

In particular,

$$g'(q^*) = u'(q^*) + \frac{\theta(1-\theta)(u(q^*) - c(q^*))(c'(q^*) - u''(q^*))}{u'(q^*)}$$

Now,

$$\begin{aligned}\lim_{m_{t+1} \nearrow m_{t+1}^*} \frac{\partial \Gamma_{t+1}^b(m_{t+1})}{\partial m_{t+1}} &= \left(\frac{u'(q^*)}{g'(q^*)} - 1 \right) \phi_{t+1} \\ &= \left(\frac{1}{1 + \frac{\theta(1-\theta)(u(q^*) - c(q^*))(c''(q^*) - u''(q^*))}{u'(q^*)^2}} - 1 \right) \phi_{t+1}\end{aligned}$$

□

Then, the first-order condition (guessing a monetary equilibrium) for money holdings is

$$\beta\alpha\sigma \left(\frac{u'(q_{t+1}(m_{t+1}))}{g'(q_{t+1}(m_{t+1}))} - 1 \right) \phi_{t+1} + \beta\phi_{t+1} - \phi_t = 0$$

or < 0 if $m_{t+1} = 0$.

Rewrite as

$$\beta\alpha\sigma\left(\frac{u'(q_{t+1}(m_{t+1}))}{g'(q_{t+1}(m_{t+1}))} - 1\right)\phi_{t+1}m_{t+1} + \beta\phi_{t+1}m_{t+1} - \phi_t m_t \underbrace{\frac{m_{t+1}}{m_t}}_{=\gamma} = 0$$

Also notice that if Γ_t^b is strictly concave, there will be a unique solution for money holdings, so that $F_t(m) = \delta_{M_t}$. Using this observation in the Euler equation, and writing $q_t = q_t(M_t)$,

$$\beta\alpha\sigma\left(\frac{u'(q_{t+1})}{g'(q_{t+1})} - 1\right)g(q_{t+1})g(q_{t+1}) + \beta g(q_{t+1}) = \gamma g(q_t) \quad (**)$$

Hence, a monetary equilibrium is a sequence $q_t \in [0, q^*]$ such that $(**)$ is satisfied

5.5 Equilibrium

Definition Given $\{M_t\}_{t=0}^\infty$, an equilibrium is a list $\{V_t, W_t, q_t, d_t, \phi_t, F_t\}_t$ satisfying

1. The Bellman equations
2. The bargaining solution
3. The first-order condition $\beta V'_{t+1}(m_{t+1}) - \phi_t m_{t+1} \leq 0$ (the Euler equation)
4. F_t degenerate at M_t for all t

Notice that, if $\gamma > \beta$, inflation is too high so that $q_t < q^*$ and there is an inefficiency.

5.6 Steady State

We look for a situation in which $q_t = q$ for all t (it is the same as saying that $\phi_t M_t$ is constant, that is, saying that $\frac{\phi_t}{\phi_{t+1}} = \frac{M_{t+1}}{M_t} = \gamma$, that is, that the quantity theory holds). Our equation becomes

$$\beta\alpha\sigma\left(\frac{u'(q)}{g'(q)} - 1\right) + \beta = \gamma$$

that is

$$\frac{u'(q)}{g'(q)} = 1 + \frac{\gamma - \beta}{\beta\alpha\sigma}$$

Recall that when $\theta = 1$, we have that $g'(q) = c'(q)$. In that case, the Friedman rule $\beta = \gamma$ is welfare-maximizing.

Exercise. Price a one-period risk-free illiquid (cannot be used in bilateral exchange) nominal bond. Let p_t be the price of such a bond. The Euler equation is

$$p_t \phi_t = \beta \phi_{t+1}$$

If $r_{t+1} = \frac{1}{p_t} - 1$ is the nominal rate, we have that $r_{t+1} = \frac{\gamma-\beta}{\beta}$. The Friedman rule, hence, is setting $r = 0$.

5.7 Welfare in Steady State

October 21, 2014

In a stationary equilibrium, $m_t = M_t$, $\phi_t M_t = g(q)$, and $V_t(m_t) = \bar{V}(q)$. Hence,

$$\begin{aligned}\bar{V}(q) &= \alpha\sigma[u(q) - g(q)] + \alpha\sigma[g(q) - c(q)] + \alpha\delta[u(q^*) - c(q^*)] + g(q) + W_t(0) \\ W_t(0) &= \phi_t \underbrace{T_t}_{\phi_{t+1}M_{t+1}-M_t} + U(X^*) - X^* + \underbrace{\max_m[-\phi_t m + \beta V_{t+1}(m)]}_{-\phi_t M_{t+1} + \beta \bar{V}(q)} \\ \implies W_t(0) &= (\gamma - 1)g(q) + U(X^*) - X^* - \gamma g(q) + \beta \bar{V}(q)\end{aligned}$$

Plugging back in,

$$(1 - \beta)\bar{V}(q) = \alpha\sigma[u(q) - c(q)] + \alpha\delta[u(q^*) - c(q^*)] + U(X^*) - X^*$$

Inflation only affects welfare through its effect on q .

Suppose the government can choose the nominal rate ($r = \frac{\gamma-\beta}{\beta}$). Its effect on welfare is

$$(1 - \beta) \frac{\partial \bar{V}(q(r))}{\partial r} = \alpha\sigma[u'(q) - c'(q)] \frac{\partial q(r)}{\partial r}$$

and

$$\frac{\partial q(r)}{\partial r} = \frac{[g'(q)]^2}{\alpha\sigma[u''(q)g'(q) - g''(q)u'(q)]} < 0$$

because $u'' < 0$ and $g', u' > 0$. The only thing that requires some assumption is showing that $g'' > 0$ (it is true if θ is close to 1 or at the equilibrium q , because of second-order conditions).

Furthermore, at marginal deviations from the Friedman rule, the welfare loss of inflation (or a higher nominal rate) is

$$\lim_{r \rightarrow 0} \frac{\partial q(r)}{\partial r} = \frac{\alpha\sigma}{\beta(1 - \beta)} (u'(q(0)) - c'(q(0))) \frac{\partial q(0)}{\partial r}$$

$(u'(q(0)) - c'(q(0))) = 0$ if $\theta = 1$ but less than zero if $\theta < 1$. When $\theta = 1$, $r = 0$ is the first-best, so the welfare loss is the first-order condition for choosing $r = 0$. When $\theta < 1$, $r = 0$ is not the first-best.

5.7.1 Measures of Welfare Cost of Inflation

1. Compensating Variation $W(r)$, defined by

$$\begin{aligned}\alpha\sigma(u(W(r)q(r)) - c(q(r))) + \alpha\delta(u(W(r)q^*) - c(q^*)) + U(W(r)X^*) - X^* \\ = \alpha\sigma(u(q(0)) - c(q(0))) + \alpha\delta(u(q^*) - c(q^*)) + U(X^*) - X^*\end{aligned}$$

2. Utility loss (per period):

$$\bar{W}(r_0, r_1) = (1 - \beta) (\bar{V}(q(r_0)) - \bar{V}(q(r_1))) = \alpha \sigma (u(q(r_0)) - c(q(r_0))) - \alpha \sigma (u(q(r_1)) - c(q(r_1)))$$

3. Area under the “money demand” curve (Bailey ’56)

5.7.2 Parametrization

$$u(q) = \frac{q^{1-\eta}}{1-\eta}, \quad \eta > 0$$

$$U(X) = B \log X, \quad B > 0$$

$$c(q) = q$$

$$\alpha = 1$$

$$\delta = 0$$

Leave free $\Pi = (\sigma, B, \eta, \beta)$ to calibrate. Notice that, under these functional forms, $q^* = 1, X^* = B$. Now, $W(r)$ solves

$$\sigma \left(\frac{(W(r)q(r))^{1-\eta}}{1-\eta} - q(r) \right) + B \log(W(r)) = \sigma \left[\frac{q(0)^{1-\eta}}{1-\eta} - q(0) \right]$$

and $q(r)$ solves

$$\frac{(\theta q^{-\eta} + 1 - \theta)^2}{\theta q^{-\eta} + (1 - \theta)(1 - \eta)} = \left(\frac{1 - \theta\eta}{1 - \eta} \right) \left(1 + \frac{r}{\alpha\sigma} \right)$$

To quantify $W(r)$, we need to calibrate the parameters in Π . For this, we’ll use four calibration targets. Two observations are in order.

First, nominal GDP: $Y_t = Y_t^D + Y_t^C = \sigma M_t + \frac{X^*}{\phi_t}$. Hence, $\frac{M_t}{Y_t} = \frac{g(q(r_t))}{B + \sigma g(q(r_t))}$

Second, this model has a markup ($g(q) > c(q)$ when $\theta < 1$): $1 + \mu_t^D = \frac{g(q(r_t))}{c(q(r_t))}$, and $1 + \mu_t^C = 1$.

Hence, the average markup is $1 + \mu_t = \frac{Y_t^D}{Y_t} (1 + \mu_t^D) + \frac{Y_t^C}{Y_t} = \frac{B + \sigma \frac{g(q(r_t))^2}{q(r_t)}}{B + \sigma g(q(r_t))}$

To estimate or calibrate Π , we do the following. First, take a time series for $L_t = \frac{M_t}{Y_t}$ and r_t from data. The model counterparts are

$$\hat{L}_t = \frac{g(q(r_t))}{B + \sigma g(q(r_t))}$$

Think of $\{\hat{L}_t\}_t$ as the fitted values to $\{L_t\}_t$. Then,

$$\Pi^* = \arg \min_{(\eta, B, \sigma, \theta)} \sum_t (\hat{L}_t(\Pi) - L_t)^2$$

$$\text{subject to } \begin{cases} \eta, B \geq 0 \\ \sigma, \theta \in [0, 1] \\ 1 + \mu_t = 1.1 \end{cases}$$

Bailey done by Lucas Inverse money demand $r = L(M/P)$. The welfare cost is $\tilde{W}^B(r_0, r_1) = \int_{(M/P)_1}^{(M/P)_0} L(z) dz$.

Suppose that $\frac{M}{Y} = Ar^{-\rho}$. Then, $L(M/Y) = \left(\frac{A}{M/Y}\right)^{\frac{1}{\rho}}$. Then,

$$\tilde{W}^B(r_0, r_1) = \int_{m^d(r_1)}^{m^d(r_0)} L(z) dz = \int_{Ar_1^{-\rho}}^{Ar_0^{-\rho}} \left(\frac{A}{z}\right)^{\frac{1}{\rho}} dz = A \frac{\rho}{1-\rho} (r_1^{1-\rho} - r_0^{1-\rho})$$

Bailey done by Lagos-Wright We can rewrite our key equation as a money demand function. Start from $\frac{u'(q)}{g'(q)} = 1 + \frac{r}{\alpha\sigma}$. If z is real balances, then $z = g(q)$, which means that $q = g^{-1}(z) = f(z)$. Hence,

$$\frac{u'(f(z))}{g'(f(z))} = 1 + \frac{r}{\alpha\sigma} \quad \rightarrow \quad z = m^d(r)$$

Inverting, we get

$$r = L(z) = \alpha\sigma \left(\frac{u'(f(z))}{g'(f(z))} - 1 \right)$$

Now,

$$\tilde{W}^B(r_0, r_1) = \int_{m^d(r_1)}^{m^d(r_0)} L(z) dz = \int_{m^d(r_1)}^{m^d(r_0)} \alpha\sigma \left(\frac{u'(f(z))}{g'(f(z))} - 1 \right) dz$$

changing variables to integrate over $q = f(z)$, we get

$$\begin{aligned} \tilde{W}^B(r_0, r_1) &= \int_{q(r_1)}^{q(r_0)} \alpha\sigma \left(\frac{u'(q)}{g'(q)} - 1 \right) \underbrace{\frac{1}{f'(q)}}_{=g'(q)} dq = \int_{q(r_1)}^{q(r_0)} \alpha\sigma (u'(q) - g'(q)) dq \\ &= \alpha\sigma (u(q(r_0)) - g(q(r_0))) - \alpha\sigma (u(q(r_1)) - g(q(r_1))) \end{aligned}$$

Almost the same as what the utility loss method yields. The only difference is that g is different from c ($\theta \neq 1$). But $\theta \neq 1$ is important in the calibration.

Now, recall that

$$g(q) = \omega(q)c(q) + (1 - \omega(q))u(q); \quad \omega(q) = \frac{\theta u'(q)}{\theta u'(q) + (1 - \theta)c'(q)}$$

We can rewrite the welfare cost as

$$\tilde{W}^B(r_0, r_1) = \alpha\sigma \omega(q(r_0)) (u(q(r_0)) - c(q(r_0))) - \alpha\sigma \omega(q(r_1)) (u(q(r_1)) - g(q(r_1)))$$

Proposition. Bailey's area under the demand curve provide a lower bound to the estimate of the welfare cost of inflation. (we will show this for small r but it is true in general) The tightness of the bound will depend on the markup

In general,

$$\tilde{W}^B(r_0, r_1) \leq \frac{\tilde{W}^B(r_0, r_1)}{\omega(q(r_0))} \leq \bar{W}(r_0, r_1)$$

Proof.

$$\omega(q(r_1)) \simeq \omega(q(r_0)) + \omega'(q(r_0))(q(r_1) - q(r_0))$$

□

Suppose that $\omega(q) = \theta$ (like in Kalai). Then, $\tilde{W}^B(r_0, r_1) = \theta \bar{W}(r_0, r_1)$. In this case, Bailey's calculation is accurate.

If $\theta = 1$, the Friedman rule is first best and Bailey's area is exact. If $\theta < 1$, $r = 0$ leaves a markup and Bailey's area is only a lower bound.

6 Liquidity and Asset Pricing

October 28, 2014

- Discrete time, 2 subperiods
- $[0, 1]$ ex-ante identical agents
- Commodities
 - Special good: subject to double coincidence of wants problem, produced during the day, nonstorable
 - General good: produced and consumed during the night by everyone, storable
- Production technology: one unit of labor gives one unit of a type of special good, or a unit of the general good
- Preferences

$$\mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t (u(q_t^b) - c(q_t^s) + y_t) \right]$$

with $y_t = Q_t - H_t$, and u strictly increasing and concave, c strictly increasing and convex. Hence, there exist unique q^* , $\bar{q}$ such that $u'(q^*) = c'(q^*)$, $u(\bar{q}) = c(\bar{q})$, the gains-from-trade maximizing and exhausting quantities.

- Storage Technology: x_t units of the general goods stored at the end of the second subperiod of t yields $k_{t+1} = F(x_t)$ units of the general good at the beginning of the first subperiod of $t + 1$. F satisfies $F(0) = 0$, $F'(0) = +\infty$, $F'' < 0$, $\lim_{k \rightarrow \infty} F'(k) < 1/\beta$.
- Assets: fiat money.
- Market Structure
 - First subperiod

- * Random matching, bilateral trade.
- * Terms of trade given by Nash bargaining. $\alpha \leq 1/2$ is the probability of a single-coincidence meeting
- * Agents may trade special goods, money, and general goods (which were stored)
- Second subperiod
 - * Walrasian trade with ϕ_t the price of a dollar in general goods
 - * Agents may trade labor (one to one to general goods), general goods, and money

6.1 The Planner's Solution

$$W = \max_{\{q_t, y_t, x_t\}_t} \sum_{t=0}^{\infty} \beta^t (\alpha [u(q_t) - c(q_t)] + y_t)$$

subject to $\begin{cases} y_t + x_t \leq k_t \\ k_{t+1} = F(x_t) \end{cases}$

Notice that $k_t = F(x_{t-1})$ in the first constraint. The first-order conditions are

$$\begin{aligned} u'(q_t) &= c'(q_t) & \implies q_t &= q^* \\ \beta F'(x_t) &= 1 & \implies x_t &=: x^* \end{aligned}$$

The planner's allocation is then (q^*, x^*, k^*) every period (where $k^* = F(x^*)$).

6.2 Bellman Equations in a Competitive Equilibrium

Let $a_t = (a_t^z, a_t^k)$ be the amount of real balances ($a_t^z = a_t^m \phi_t$) and the amount of 'capital' the agent holds, respectively.

Let $W_t(a_t)$ be the value of entering the competitive round with holdings a_t . Then,

$$W(a_t) = \max_{y_t, x_t^m, x_t^k} y_t + \beta V(\bar{a}_{t+1})$$

subject to $\begin{cases} y_t + \phi_t x_t^m + x_t^k = a_t^z + a_t^k + \tau_t \\ \bar{a}_{t+1}^k = F(x_t^k) \\ \bar{a}_{t+1}^z = \phi_{t+1} x_t^m \end{cases}$

where $\bar{a}_t$ is the portfolio that the agent holds, and a_t is the portfolio that the agent brings to the bargaining table.

Rewriting, and using real balances as the choice (and specializing to a steady-state in which $M_{t+1}/M_t = \phi_t/\phi_{t+1} = \gamma$)

$$W(a_t) = a_t^z + a_t^k + \tau_t + \max_{x_t^z, x_t^k} \left\{ -x_t^z - x_t^k + \beta V \left(\frac{x_t^z}{\gamma}, F(x_t^k) \right) \right\}$$

Notice that $W(a_t) = a_t^z + a_t^k + W(0, 0)$.

Also, there is this problem with gains from trade not being monotonic in wealth. Hence, we will allow agents to not necessarily bring everything they have to the bargaining table. In that case,

$$\begin{aligned} V(\bar{a}_t) = \max_{a_t \leq \bar{a}_t} \Big\{ & \alpha \int \{ u(q(a_t, \tilde{a}_t)) + W(\bar{a}_t - d(a_t, \tilde{a}_t)) - W_t(\bar{a}_t) \} dH(\tilde{a}_t) \\ & + \alpha \int \{ -c(q(\tilde{a}_t, a_t)) + W(\bar{a}_t + d(\tilde{a}_t, a_t)) - W_t(\bar{a}_t) \} dH(\tilde{a}_t) \\ & + W(\bar{a}_t) \Big\} \end{aligned}$$

Hence,

$$\begin{aligned} V(\bar{a}_t) = \max_{a_t \leq \bar{a}_t} \Big\{ & \alpha \int \{ u(q(a_t, \tilde{a}_t)) - (d^z(a_t, \tilde{a}_t) + d^k(a_t, \tilde{a}_t)) \} dH(\tilde{a}_t) \\ & + \alpha \int \{ -c(q(\tilde{a}_t, a_t)) + (d^z(\tilde{a}_t, a_t) + d^k(\tilde{a}_t, a_t)) \} dH(\tilde{a}_t) \\ & + W(a_t) \Big\} \end{aligned}$$

6.3 Determination of q and d

When buyer and seller meet, they carry a and $\tilde{a}$, respectively. The Nash bargaining problem is

$$\begin{aligned} [q(a, \tilde{a}), d(a, \tilde{a})] = \arg \max_{q, d} [u(q) + W(a - d) - W(a)]^\theta [-c(q) + W(\tilde{a} + d) - W(\tilde{a})]^{1-\theta} \\ \text{subject to } (d^z, d^k) \leq (a^z, a^k) \end{aligned}$$

Lemma. Let $g(q) = \frac{(1-\theta)c'(q)u(q) + \theta u'(q)c(q)}{(1-\theta)c'(q) + \theta u'(q)}$

- If $a^z + a^k \geq g(q^*)$, then $q = q^*$ and (d^z, d^k) is any pair that solves $d^z + d^k = g(q^*)$
- If $a^z + a^k < g(q^*)$, then q solves $g(q) = a^z + a^k$ and the buyer gives up his whole portfolio ($d = a$)

We can rewrite this more compactly as $v = a^z + a^k$ and then

$$\begin{aligned} q(v) = \begin{cases} q^* & \text{if } v \geq g(q^*) \\ g^{-1}(v) & \text{if } v < g(q^*) \end{cases} \\ d^z + d^k = g[q(v)] \end{aligned}$$

Plug back into the value function to get

$$V(\bar{a}_t) = \max_{v_t \leq \bar{a}_t} \left\{ \alpha \{u(q(v_t)) - g(q(v_t))\} + \alpha \int \{-c(q(\tilde{v}_t)) + g(q(\tilde{v}_t))\} dF(\tilde{v}_t) + W(\bar{a}_t) \right\}$$

Or,

$$V(\bar{a}_t) = \max_{v_t \leq \bar{a}_t^k + \bar{a}_t^z} [\alpha S_b(v_t) + \alpha S_s + W(\bar{a}_t)]$$

where $s_b(v_t) = u[q(v_t)] - g[q(v_t)]$ and $S_s = \int \{g[q(\tilde{v})] - c[q(\tilde{v})]\} dF(\tilde{v})$.

Let $q_b^* = \arg \max_q [u(q) - g(q)]$. Then,

$$V(\bar{a}_t) = \begin{cases} \alpha [u(q_b^*) - g(q_b^*)] + \alpha S_s + W(\bar{a}_t) & \text{if } \bar{v}_t \geq g(q_b^*) \\ \alpha [u(q(v_t)) - g(q(v_t))] + \alpha S_s + W(\bar{a}_t) & \text{if } \bar{v}_t < g(q_b^*) \end{cases}$$

Or,

$$V\left(\frac{x_t^z}{\gamma}, F(x_t^k)\right) = \begin{cases} \alpha [u(q_b^*) - g(q_b^*)] + \alpha S_s + W(\bar{a}_t) & \text{if } \bar{v}_t \geq g(q_b^*) \\ \alpha [u(q(v_t)) - g(q(v_t))] + \alpha S_s + W(\bar{a}_t) & \text{if } \bar{v}_t < g(q_b^*) \end{cases}$$

Going back,

$$W(a_t) = a_t^z + a_t^k + \tau_t + \max_{x_t^z, x_t^k} \left\{ x_t^z - x_t^k + \beta V\left(\frac{x_t^z}{\gamma}, F(x_t^k)\right) \right\}$$

Looking at the maximization problem, using that $W(\bar{a}_t) = \bar{v}_t + W(0, 0)$, we have

$$\max_{x_t^z, x_t^k} \left\{ \beta \left[\alpha \left\{ u \left[\tilde{q} \left(\frac{x_t^z}{\gamma} + F(x_t^k) \right) \right] - g \left[\tilde{q} \left(\frac{x_t^z}{\gamma} + F(x_t^k) \right) \right] \right\} + \frac{x_t^z}{\gamma} + F(x_t^k) \right] - x_t^z - x_t^k \right\}$$

where

$$\tilde{q}(v) = \begin{cases} q_b^* & \text{if } v \geq g(q_b^*) \\ g^{-1}(v) & \text{if } v < g(q_b^*) \end{cases}$$

Taking first-order conditions,

$$\begin{aligned} \frac{\beta}{\gamma} L(z_t) &\leq 1, & \text{if } x_t^z > 0 \\ \beta F'(x_t^k) L(z_t) &\leq 1, & \text{if } x_t^k > 0 \end{aligned}$$

where $z_t = \frac{x_t^z}{\gamma} + F(x_t^k)$ and $L(z_t) = 1 + \alpha \left\{ \frac{u'[\tilde{q}(z_t)]}{g'[\tilde{q}(z_t)]} - 1 \right\}$.

6.4 Equilibrium

Let $k = F(x^k)$

Proposition. There exists a unique nonmonetary equilibrium. Further,

1. if $k^* \geq g(q_b^*)$, the nonmonetary equilibrium has $k = k^*$, $q = q_b^*$, $L(k) = 1$, for k^* the planner's choice of capital.
2. If $k^* < g(q_b^*)$, then $k > k^*$, $q < q_b^*$, and $L(k) > 1$: more capital, less consumption, and a liquidity premium on capital.

Proposition. There exists $\bar{\gamma} > \beta$ such that a unique (stationary) monetary equilibrium exists if $k^* < g(q_b^*)$ and $\gamma \in [\beta, \bar{\gamma}]$. Further,

1. $\frac{\partial k}{\partial \gamma} > 0$
2. $\frac{\partial q}{\partial \gamma} < 0$ and therefore $\frac{\partial W}{\partial \gamma} < 0$
3. As $\gamma \rightarrow \bar{\gamma}$, the allocation converges to the nonmonetary equilibrium
4. As $\gamma \rightarrow \beta$, $q \rightarrow q_b^*$.

Proof. In a monetary equilibrium, from the first-order conditions, we get that $\beta L = \gamma$ and therefore $F'(x^k) = 1/\gamma$: because both assets have the same liquidity, can be traded at the same times, etc, they have to give the same return. Hence, $\frac{\partial x^k}{\partial \gamma} > 0$.

Also, $\frac{\partial v}{\partial \gamma} < 0$: inflation makes real balances less attractive, tilting the agent's portfolio toward the real asset, but decreasing the total wealth held.

We only need to find conditions that ensure that $x^z > 0$ for the FOCs to hold with equality. Let $\Gamma(\gamma) = v - F(x^k)$, notice that $\Gamma'(\gamma) < 0$. For a monetary equilibrium, we need that $\Gamma(\gamma) > 0$ so that a positive proportion of wealth is held in money. Hence, a monetary equilibrium exists if $\Gamma(\beta) > 0$. In that case, implicitly define $\Gamma(\bar{\gamma}) = 0$.

Finally, we need to transform $\Gamma(\beta) > 0$ into a condition on primitives. If $\gamma = \beta$, we have that $L(v) = 1$ by the FOCs, which means that $\beta F'(x^k) = 1$, that is, $x^k = x^*$. Hence, v has to be the one that supports q_b^* , that is $v = g(q_b^*)$. In this case, $\Gamma(\beta) = g(q_b^*) - k^* > 0$. $\square$

When capital alone cannot support q_b^* , the nonmonetary equilibrium over-accumulates it. Money is valued only while its return stays close to β , and it crowds that capital back down.

6.5 Example (Wallace '80)

$F(x) = Ax$ with $A \geq 0$.

	Planner	Non Monetary Equilibrium	Monetary Equilibrium
$\beta A > 1$	FOC: $\beta A \leq 1$ No solution	FOC: $\beta AL(F(x^n)) \leq 1$ No equilibrium	FOC: $\frac{\beta}{\gamma} L(v) \leq 1; \beta AL(v) \leq 1$ No equilibrium
$\beta A = 1$	$\text{sol } x^* \geq 0$	$L = 1$, so $q = q_b^*$, $F(x^n) \geq g(q_b^*)$ but x^n indeterminate	$L(v) = 1$, which needs $\gamma = \beta$ x^k indeterminate (assets are equal)
$\beta A < 1$	$\text{sol } x^* = 0$	$\implies L > 1$, so $q^n < q_b^*$ q^n solves $\frac{u'(q)}{g'(q)} = 1 + \frac{1-\beta A}{\alpha \beta A}$ $Ax^n = g(q^n) \implies x^n > 0$	$x^z > 0$ needs $\beta L(v) = \gamma$ $\implies \gamma \in [\beta, \frac{1}{A})$ if $\gamma > \beta$, $x^m = 0 = x^* < x^n$ $\frac{u'(q^n)}{g'(q^n)} = 1 + \frac{\gamma-\beta}{\alpha \beta}$ $q^n < q^m \leq q_b^*$, = if $\gamma = \beta$.

7 Liquidity and Lucas Trees

November 4, 2014

- Back to nonstorable general good
- Stock A of Lucas Trees. Each tree yields a dividend y of general goods, in the second subperiod
- Equity shares on the trees are tradable.
- Simplification: $c(q) = q$ and $\theta = 1$.
- Assume that $\beta \left[1 + \alpha \left\{ u' \left(\frac{\gamma A}{1-\beta}, q^* \right) - 1 \right\} \right] < 1$.¹ Also, $U(Q) = Q$ as before.

7.1 The Planner's Problem

A Planner solves

$$\mathcal{W} = \max_{q_t, Q_t} \sum_t \beta^t \{ \alpha [u(q_t) - c(q_t)] + Q_t \}$$

subject to $Q_t \leq yA$

The solution has $q_t = q^*$, $Q_t = yA$.

7.2 Equilibrium

Let $W_t(m_t, a_t)$ be the value of entering the second subperiod with m_t coins and a_t shares, and $V_t(m_t, a_t)$ the value of entering the first subperiod. Hence,

$$W_t(m_t, a_t) = \max_{Q_t, m_{t+1}, a_{t+1}} \{ Q_t + \beta V_{t+1}(m_{t+1}, a_{t+1}) \}$$

$$\text{subject to } Q_t + \phi_t^m m_{t+1} + \phi_t^s a_{t+1} = \phi_t^m m_t + \underbrace{(y + \phi_t^s)}_{=\bar{\phi}_t^s} a_t + \tau_t$$

¹this will be a stochastic discount factor

or,

$$\begin{aligned} W_t(m_t, a_t) &= \phi_t^m m_t + \bar{\phi}_t^s a_t + \tau_t + \max_{m_{t+1}, a_{t+1}} \{-\phi_t^m m_{t+1} - \phi_t^s a_{t+1} + \beta V_{t+1}(m_{t+1}, a_{t+1})\} \\ &= \phi_t^m m_t + \bar{\phi}_t^s a_t + W_t(0, 0) \end{aligned}$$

In the first subperiod, if $x_t = (m_t, a_t)$,

$$\begin{aligned} V_t(x_t) &= \alpha \int \{u[q_t(x_t, \tilde{x}_t)] + W_t[x_t - d_t(x_t, \tilde{x}_t)]\} dF_t(\tilde{x}_t) \\ &\quad + \alpha \int \{-q_t(\tilde{x}_t, x_t) + W_t(\tilde{x}_t + d_t(\tilde{x}_t, x_t))\} dF_t(\tilde{x}_t) \\ &\quad + (1 - 2\alpha)W_t(x_t) \end{aligned}$$

Under Nash bargaining with $\theta = 1$, the buyer solves

$$\begin{aligned} [q_t(x_t, \tilde{x}_t), d_t(x_t, \tilde{x}_t)] &= \arg \max_{q_t, d_t} \{[u(q_t) + W_t(x_t - d_t) - W_t(x_t)]\} \\ \text{subject to } &\begin{cases} q_t + W_t(\tilde{x}_t + d_t) \geq W_t(\tilde{x}_t) \\ \underbrace{d_t}_{(d_t^m, d_t^s)} \leq (m_t, a_t) \end{cases} \end{aligned}$$

Lemma. The solution to the bargaining problem is

$$q(z_t) = \phi_t^m d_t^m + \bar{\phi}_t^s d_t^s = \min(z_t, q^*)$$

where $z_t = \phi_t^m m_t + \bar{\phi}_t^s a_t$

Manipulating the Bellman equation for V gives

$$\begin{aligned} V_t(x_t) &= \alpha \int \left\{ u[q_t(x_t, \tilde{x}_t)] + \underbrace{\phi_t^m d_t^m + \bar{\phi}_t^s d_t^s}_{=-q(z_t)} \right\} dF_t(\tilde{x}_t) + W_t(x_t) \\ &= \alpha \{u[q(z_t)] - q(z_t)\} + z_t + W(0, 0) \end{aligned}$$

where $z_t = \phi_t^m m_t + \bar{\phi}_t^s a_t$.

Plugging back into W , we get $W_t(z_t) = z_t + W_t(0, 0)$ where

$$W_t(0, 0) = \max_{m_{t+1}, a_{t+1}} \{-\phi_t^m m_{t+1} - \phi_t^s a_{t+1} + \beta V_{t+1}(m_{t+1}, a_{t+1})\}$$

Hence,

$$\begin{aligned}
\beta \frac{\partial V_{t+1}}{\partial m_{t+1}}(m_{t+1}, a_{t+1}) &\leq \phi_t^m \quad (= \text{ if } m_{t+1} > 0) \\
\beta \frac{\partial V_{t+1}}{\partial a_{t+1}}(m_{t+1}, a_{t+1}) &= \phi_t^s \\
\frac{\partial V_t}{\partial m_t}(m_t, a_t) &= \left\{ 1 + \alpha \{u'[q(z_t)] - 1\} \frac{\partial q}{\partial m_t}(z_t) \frac{1}{\phi_t^m} \right\} \phi_t^m \\
&= \underbrace{[1 + \alpha \{u'[\min(z_t, q^*)] - 1\}]}_{=L(z_t)} \phi_t^m
\end{aligned}$$

going back,

$$\begin{aligned}
\beta L(z_{t+1}) \phi_{t+1}^m &\leq \phi_t^m \quad (= \text{ if } m_{t+1} > 0) \\
\beta L(z_{t+1}) (\phi_{t+1}^s + y) &= \phi_t^s
\end{aligned}$$

We'll look at stationary equilibrium, nonstationary ones include sunspots, market crashes etc (see Rocheteau and Wright, JME 2012).

7.3 Nonmonetary equilibrium

$$(z = z^m + z^s = z^s = (\phi^s + y)A)$$

The relevant Euler equation is

$$\beta L(z)(\phi^s + y) = \phi^s$$

Proposition. There exists a nonmonetary equilibrium. Let $\phi_s^* = \frac{\beta}{1-\beta}y$ (the non-liquidity price of a tree). Then,

- a. If $q^* \leq \frac{1}{1-\beta}yA$, (liquidity is not scarce) then $q = q^*$ in every meeting, $\phi^s = \phi_s^*$, and $L(z) = L(\phi_s^*A) = 1$.
- b. If, otherwise, $q^* > \frac{1}{1-\beta}yA$, (there is a liquidity shortage) then $q < q^*$, ϕ^s solves the Euler equation $\beta L[(\phi^s + y)A](\phi^s + y) = \phi^s$, and $L(z) > 1$.

7.4 Monetary equilibrium

$$(M_{t+1}/M_t = \gamma), z = \underbrace{z^m}_{=\phi_t^m M_t} + \underbrace{(\phi^s + y)A}_{z^s}$$

The Euler equations are

$$\begin{aligned}
\beta L(z) &= \gamma \\
\beta L(z)(y + \phi^s) &= \phi^s
\end{aligned}$$

Proposition. A monetary equilibrium exists provided that

- $\frac{1}{1-\beta}yA < q^*$, that is, there is a liquidity shortage
- $\gamma \in [\beta, \bar{\gamma})$, where $\bar{\gamma} < 1 - \frac{yA}{q^*}$ and solves

$$\beta \left[1 + \alpha \left\{ u' \left(\min \left\{ \frac{yA}{1-\bar{\gamma}}, q^* \right\} - 1 \right) \right\} \right] - \bar{\gamma} = 0$$

Moreover,

- The equity price equals $\phi^s = \frac{\gamma}{1-\gamma}y$, the liquidity premium is $L(z) = \frac{\gamma}{\beta}$, and $q \leq q^*$ (= if $\gamma = \beta$)
- As $\gamma \rightarrow \beta$, $\phi^s \rightarrow \phi_s^*$, $L(z) \rightarrow 1$, and $q \rightarrow q^*$.
- As $\gamma \rightarrow \bar{\gamma}$, prices and allocations converge to those of the nonmonetary equilibrium.

At the Friedman rule, there is no liquidity premium and the tree's price returns to its fundamental value ϕ_s^* .

8 Over-The-Counter Markets

November 11, 2014

8.1 Market Microstructure

- Order-driven markets: NYSE. Intermediaries auction assets
- Quote-driven markets: Nasdaq. Dealers are required to post and honor bid and ask prices
- OTC: derivatives, treasuries, real assets

Jargon Dealers (become a counterparty, as car dealers), market-makers, brokers (only help you to match), specialists (auctioneers at NYSE), ECNs (electronic communication networks)

Reference Duffie, Garleanu and Pedersen (2005) *Econometrica*

8.2 Environment

- Continuous time, infinite horizon
- Two types of agents
 - Investors
 - Dealers
- Commodities and Assets

- Durable assets in supply A
- Consumption good which also serves as numeraire
- Asset inventories $i \in \{0, 1\}$
- Preferences: Everyone discounts at the rate $r > 0$.
 - Investors: u_{ij} is the flow utility to an investor who holds i assets and has valuation j , $j \in \{0, 1\}$
- We maintain the assumption that $u_{11} + u_{00} - u_{01} - u_{10} > 0$, that is, from a myopic point of view, there is a joint gain from trade between two investors with different holdings (notice that this is consistent with $0 = u_{00} = u_{01} < u_{10} < u_{11}$).
- Market Structure:
 - Investors find investors at rate β , a dealer finds a dealer at rate α , and there is a price p for the asset in the interdealer market
 - Investors meet dealers at rate α . The dealer is forced to trade at her bid/ask price, and they use p^A or p^B depending on which side they are on

8.3 Bellman Equations

Let V_{ij} be the maximized expected value of holding i assets when valuation is j . Let $\delta_u = \delta\pi$ and $\delta_d = \delta(1 - \pi)$. Then,

$$\begin{aligned}
 rV_{11} &= u_{11} + \delta_d (V_{10} - V_{11}) \\
 rV_{00} &= u_{00} + \delta_u (V_{01} - V_{00}) \\
 rV_{01} &= u_{01} + \alpha\phi^I\phi^0(V_{11} - V_{01} - p^A) + \beta m_{10}\chi^H\chi^L(V_{10} - V_{01} - p^I) + \delta_d(V_{00} - V_{01}) \\
 rV_{10} &= u_{10} + \alpha\psi^I\psi^D(V_{00} - V_{10} + p^B) + \beta m_{01}\chi^L\chi^H(V_{00} - V_{10} + p^I) + \delta_u(V_{11} - V_{10})
 \end{aligned}$$

8.4 Bargaining

1. Investor-Investor meetings (p^I)

Assume high-value investor has bargaining power θ . Then, the Nash Bargaining problem is

$$\max_p (V_{11} - V_{01} - p)^\theta (V_{00} - V_{10} + p)^{1-\theta}$$

which gives

$$p^I = \theta(V_{10} - V_{00}) + (1 - \theta)(V_{11} - V_{01})$$

Call $S = V_{11} + V_{00} - V_{01} - V_{10}$. Then the gains to the buyer are θS and the gains to the seller are $(1 - \theta)S$. Also, there is trade iff $S \geq 0$, so that $\chi^H\chi^L = \mathbb{1}(S \geq 0)$.²

²Really, $\chi^H\chi^L$ is allowed to be anything in $[0, 1]$ when $S = 0$

2. Investor-Dealer meetings when the investor buys (p^A)

If the dealer has bargaining power η , the Nash Bargaining problem is

$$\max_{p^A} (V_{11} - V_{01} - p^A)^{1-\eta} (p^A - p)^\eta$$

which yields $p^A = \eta(V_{11} - V_{01}) + (1 - \eta)p$. The gains from trade to the buyer are then $V_{11} - V_{01} - p^A = (1 - \eta)(V_{11} - V_{01} - p)$ and those to the seller are $p^A - p = \eta(V_{11} - V_{01} - p)$. As before, both parties agree on when trade should take place, so that $\phi^I \phi^D = \mathbb{1}(V_{11} - V_{01} - p \geq 0)$.

3. Investor-Dealer meetings when the investor sells (p^B)

$$\max_{p^B} (p^B + V_{00} - V_{10})^{1-\eta} (p - p^B)^\eta$$

which yields $p^B = \eta(V_{10} - V_{00}) + (1 - \eta)p$, so that $\psi^I = \psi^D = \mathbb{1}(p + V_{00} - V_{10} \geq 0)$

8.5 Law of Motion for Asset Holdings

Obs

1. $n_{11} + n_{01} + n_{00} + n_{10} = 1$ because they sum up to the total population
2. $n_{11} + n_{10} = A$ because each agent who holds holds one unit

We have that

$$\begin{aligned} \dot{n}_{11} &= \underbrace{\delta_u n_{10}}_{\text{inflow}} - \underbrace{\delta_d n_{11}}_{\text{outflows}} + \underbrace{\beta \chi n_{10} n_{01}}_{\text{trading}} + \underbrace{\alpha n_s}_{\text{possible rationing}} \\ \dot{n}_{00} &= \delta_d n_{01} - \delta_u n_{00} + \beta \chi n_{10} n_{01} + \alpha n_s \\ \dot{n}_{01} &= \delta_u n_{00} - \delta_d n_{01} - \beta \chi n_{01} n_{10} - \alpha n_s \\ \dot{n}_{10} &= \delta_d n_{11} - \delta_u n_{10} - \beta \chi n_{10} n_{01} - \alpha n_s \end{aligned}$$

where $n_s = \min\{\psi n_{10}, \phi n_{01}\}$

DGP has existence and uniqueness of steady state.

Now, let $m_1 = n_{01} + n_{11}$ and $m_0 = n_{00} + n_{10}$ be the measures of investors with high and low valuation. Then, $\dot{m}_1 = \delta_u m_0 - \delta_d m_1$ and $\dot{m}_0 = \delta_d m_1 - \delta_u m_0$. Also, $m_0 + m_1 = 1$ and $\pi m_0 = (1 - \pi) m_1$, which implies that $m_1 = \pi$.

Therefore, $n_{10} - n_{01} = A - \pi$.

Now, market clearing in the interdealer market means $\psi n_{10} = \phi n_{01}$, which pins down p .

Guessing configurations for ϕ and ψ gives that

$$p \begin{cases} = V_{11} - V_{01} & \text{if } A < \pi \\ \in (V_{10} - V_{00}, V_{11} - V_{01}) & \text{if } A = \pi \\ = V_{10} - V_{00} & \text{if } A > \pi \end{cases}$$

The interdealer price sits at a reservation value unless $A = \pi$. The bid–ask spread is the dealer’s share of the same surplus investors could split with each other.

Taking this, along with the terms of trade and the best-responses, back to the Bellman equations, we get

$$\begin{aligned} rV_{11} &= u_{11} + \delta_d(V_{10} - V_{11}) \\ rV_{00} &= u_{00} + \delta_u(V_{01} - V_{00}) \\ rV_{01} &= u_{01} + \delta_d(V_{00} - V_{01}) + [\alpha\epsilon(1 - \eta) + \beta n_{10}\theta] (V_{11} + V_{00} - V_{01} - V_{10}) \\ rV_{10} &= u_{10} + \delta_u(V_{11} - V_{10}) + [\alpha(1 - \epsilon)(1 - \eta) + \beta n_{01}(1 - \theta)] (V_{11} + V_{00} - V_{01} - V_{10}) \end{aligned}$$

which gives an expression for surplus

$$V_{11} + V_{00} - V_{10} - V_{01} = \frac{u_{11} + u_{00} + u_{10} + u_{01}}{r + \delta + \alpha(1 - \eta) + \beta[\theta n_{10} + (1 - \theta)n_{01}]}$$

Finally, the prices are

$$\begin{aligned} p^I &= \theta(V_{10} - V_{00}) + (1 - \theta)(V_{11} - V_{01}) \\ p &= \epsilon(V_{10} - V_{00}) + (1 - \epsilon)(V_{11} - V_{01}) \\ p^A &= (1 - \eta)p + \eta(V_{11} - V_{01}) = \omega(V_{10} - V_{00}) + (1 - \omega)(V_{11} - V_{01}) \\ p^B &= \eta(V_{10} - V_{00}) + (1 - \eta)p = \gamma(V_{11} - V_{01}) + (1 - \gamma)(V_{10} - V_{00}) \end{aligned}$$

where $\omega = (1 - \eta)\epsilon$ and $\gamma = (1 - \eta)(1 - \epsilon)$.

With $\beta = 0$, we can dispense with asset holdings in $\{0, 1\}$

November 18, 2014

9 OTC meets Mortensen-Pissarides (Lagos-Rocheteau, 2009)

9.1 Environment

- Continuous time and infinite horizon
- Population:
 - Dealers in $[0, N]$ (for now $N = 1$)

- Investors in $[0, 1]$
- Divisible and storable assets in a set A
- Commodities: general good
- Preferences: discount rate of $r > 0$
 - Investors: $u_i(a) + c$ where c is net consumption of the general good and a is asset holdings and $i \in X = \{1, \dots, I\}$ is a preference shock (switches arrive with Poisson rates and the new state is drawn *iid* independent of current state). $u : X \times \mathbb{R}_+ \rightarrow \mathbb{R}$ is concave.
 - Dealers get instantaneous utility by consuming the general good ($u_d(a, c) = c$)
- Market structure: interdealer market price p_t , investors meet dealers with Poisson rate α .

9.2 Bellman Equations

For an investor of type i who holds assets a at time t , we have

$$V_i(a, t) = \mathbb{E}_i \left[\int_t^{T_\alpha} e^{-r(s-t)} u_{k(s)}(a) ds + e^{-r(T_\alpha-t)} \left\{ V_{k(T_\alpha)} [a_{k(T_\alpha)}(T_\alpha), T_\alpha] - p(T_\alpha) [a_{k(T_\alpha)}(T_\alpha) - a] - \phi_{k(T_\alpha)}(a, T_\alpha) \right\} \right]$$

where T_α is the time of the first contact with a dealer, $k(s)$ is the preference type at time s

10 Fed Funds markets (Afonso and Lagos)

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